

On pairwise sensitive homeomorphisms

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Abstract

We obtain properties of the pairwise sensitive homeomorphisms defined in [3]. For instance, we prove that their sets of points with converging semi-orbits have measure zero, that such homeomorphisms do not exist in a compact interval and, in the circle, they are the Denjoy ones. Applications including alternative proofs of well-known facts in expansive systems are given.

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1 Introduction

Let (X, d) a compact metric space and μ be a Borel probability measure of X with induced product measure $\mu^{\otimes 2}$. A homeomorphism $f : X \rightarrow X$ is *pairwise sensitive*, with respect to μ , if there is positive constant δ such that for $\mu^{\otimes 2}$ -a.e. $(x, y) \in X \times X$ there is $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \delta$. This definition was introduced in [3] although in the more general context of non-invertible maps. Sufficient conditions for pairwise sensitivity were then obtained as, for instance, to be weakly mixing or ergodic with positive entropy on measurable partitions formed by continuity sets (c.f. [3]).

In this paper we obtain some properties of the pairwise sensitivity homeomorphisms on compact metric spaces. For instance, we prove that the set of points with converging semi-orbits under these homeomorphisms has measure zero. Additionally, there are no such homeomorphisms of compact intervals and, in the circle, they are precisely the Denjoy ones. These results allow us to obtain probabilistic proofs of some well-known results about expansive systems.

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2 Statements and proofs

Given a homeomorphism $f : X \rightarrow X$, $x \in X$ and $\delta > 0$ we define the dynamic ball

$$\Gamma_\delta(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \delta, \forall n \in \mathbb{Z}\}.$$

(The notation $\Gamma_\delta^f(x)$ to indicate the dependence on f .)

We start with the following characterization of the pairwise sensitive homeomorphisms.

Lemma 2.1 *The following properties are equivalent for every homeomorphism f of a metric space X :*

- (a) f is pairwise sensitive.
- (b) There is $\delta > 0$ such that

$$\mu(\Gamma_\delta(x)) = 0, \quad \forall x \in X. \quad (1)$$

- (c) There is $\delta > 0$ such that

$$\mu(\Gamma_\delta(x)) = 0, \quad \forall \mu\text{-a.e. } x \in X. \quad (2)$$

PROOF. Given $\delta > 0$ define \mathcal{A}_δ and $f \times f : X \times X \rightarrow X \times X$ by

$$\mathcal{A}_\delta = \{(x, y) \in X \times X : d(x, y) \leq \delta\} \quad \text{and} \quad (f \times f)(x, y) = (f(x), f(y)).$$

As noticed in [3], f is pairwise sensitive if and only if there is $\delta > 0$ satisfying

$$\mu^{\otimes 2} \left(\bigcap_{n \in \mathbb{N}} (f \times f)^{-n}(\mathcal{A}_\delta) \right) = 0. \quad (3)$$

On the other hand, the following inequalities hold

$$\bigcap_{n \in \mathbb{N}} (f \times f)^{-n}(\mathcal{A}_\delta) \subset \bigcup_{x \in X} (\{x\} \times \Gamma_\delta(x)) \subset \bigcap_{n \in \mathbb{N}} (f \times f)^{-n}(\mathcal{A}_{2\delta})$$

so

$$F_\delta(x, y) \leq \chi_{\Gamma_\delta(x)}(y) \leq F_{2\delta}(x, y),$$

where F_δ and χ_C denotes the characteristic functions of $\bigcap_{n \in \mathbb{N}} (f \times f)^{-n}(\mathcal{A}_\delta)$ and $C \subset X$ respectively. Integrating the last expression we obtain

$$\mu^{\otimes 2} \left(\bigcap_{n \in \mathbb{N}} (f \times f)^{-n}(\mathcal{A}_\delta) \right) \leq \int_X \int_X \chi_{\Gamma_\delta(x)}(y) d\mu(y) d\mu(x)$$

$$\leq \mu^{\otimes 2} \left(\bigcap_{n \in \mathbb{N}} (f \times f)^{-n}(\mathcal{A}_{2\delta}) \right) \quad (4)$$

Now suppose that (a) holds, i.e., f is pairwise sensitive. So, there is $\delta > 0$ satisfying (3). Then, the second inequality in (4) implies $\mu(\Gamma_{\frac{\delta}{2}}(x)) = 0$ for μ -a.e. $x \in X$ whence (c) holds. On the other hand, if (b) holds, i.e., if there is $\delta > 0$ satisfying (1), then the first inequality in (4) implies (3) so f is pairwise sensitive proving (a).

Next we prove that (b) and (c) are equivalent. Indeed, we only have to prove that (c) implies (b). Suppose by contradiction that (c) holds but not (b). Since (c) holds there is $\delta > 0$ satisfying (2), and, since (b) fails, there is $x_0 \in X$ such that $\mu(\Gamma_{\delta/2}(x_0)) > 0$. Denote $X_\delta = \{x \in X : \mu(\Gamma_\delta(x)) = 0\}$ so $\mu(X_\delta) = 1$. Since μ is a probability we obtain $X_\delta \cap \Gamma_{\frac{\delta}{2}}(x_0) \neq \emptyset$ so there is $y_0 \in \Gamma_{\frac{\delta}{2}}(x_0)$ such that $\mu(\Gamma_\delta(y_0)) = 0$. Now if $x \in \Gamma_{\frac{\delta}{2}}(x_0)$ we have $d(f^i(x), f^i(x_0)) \leq \frac{\delta}{2}$ (for all $i \in \mathbb{N}$) and, since $y_0 \in \Gamma_{\frac{\delta}{2}}(x_0)$, we obtain $d(f^i(y_0), f^i(x_0)) \leq \frac{\delta}{2}$ (for all $i \in \mathbb{N}$) so $d(f^i(x), f^i(y_0)) \leq d(f^i(x), f^i(x_0)) + d(f^i(x_0), f^i(y_0)) \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta$ (for all $i \in \mathbb{N}$) proving $x \in \Gamma_\delta(y_0)$. Therefore $\Gamma_{\frac{\delta}{2}}(x_0) \subset \Gamma_\delta(y_0)$ so $\mu(\Gamma_{\frac{\delta}{2}}(x_0)) \leq \mu(\Gamma_\delta(y_0)) = 0$ which is absurd. This proves that (b) and (c) are equivalent. This proves the result. \square

For the next remark we recall the classical definition of expansive homeomorphism [10]:

Definition 2.2 *A homeomorphism f is expansive if there is a positive δ (called expansivity constant) such that $x = y$ whenever $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$.*

Remark 2.3 *It follows from the definition that a homeomorphism is expansive if and only if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in X$. This suggests to define measure-expansive homeomorphism as a homeomorphism f for which there is a positive δ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$ (see the unpublished paper [8]). However, Lemma 2.1 implies that this definition is in fact equivalent to pairwise sensitivity. Nevertheless, it is still pertinent to compare the pairwise sensitive and expansive systems.*

The following definition is motivated by the definition of expansivity constant above.

Definition 2.4 *A μ -expansivity constant of f is a constant δ satisfying either (b) or (c) in Lemma 2.1.*

It follows that f is pairwise sensitive with respect to μ if and only if it has a

μ -expansivity constant. From this we obtain the following corollary which can be also derived directly from the definition.

Corollary 2.5 *Every expansive homeomorphism is pairwise sensitive with respect to any nonatomic Borel probability measure.*

Notice that the converse of this corollary is false. More precisely, there are homeomorphisms on certain compact metric spaces which are pairwise sensitive with respect to any nonatomic Borel probability measure in X_0 but not expansive. As a first example we can mention the identity $f : X_0 \rightarrow X_0$ in $X_0 = \{\frac{1}{n} : n \in \mathbb{N}^+\} \cup \{0\}$ with the metric induced by \mathbb{R} . Indeed since there is no nonatomic Borel probability measures in X_0 , f is automatically pairwise sensitive with respect to every nonatomic Borel probability measure but it is not expansive (the author is indebted to a referee for suggesting this example). A second example in which X_0 is a Cantor set is given in [7].

The following is nothing but a pairwise sensitive version of a well-known characterization of the expansive homeomorphisms.

Proposition 2.6 *If $n \in \mathbb{Z} \setminus \{0\}$, then a homeomorphism f is pairwise sensitive if and only if f^n is.*

PROOF. It follows trivially from the definition of pairwise sensitivity that if f^n is pairwise sensitive, then so is f . Conversely, suppose that f is pairwise sensitive, and so, it has a μ -expansivity constant δ . Since X is compact and n is fixed we can choose $0 < \epsilon < \delta$ such that if $d(x, y) \leq \epsilon$, then $d(f^i(x), f^i(y)) < \delta$ for all $-n \leq i \leq n$. With this property one has $\Gamma_\epsilon^{f^n}(x) \subset \Gamma_\delta^f(x)$ for all $x \in X$ yielding $\mu(\Gamma_\epsilon^{f^n}(x)) = 0$ for μ -a.e. $x \in X$. Thus f^n is pairwise sensitive by Lemma 2.1. \square

If f is a homeomorphism of X , and $z \in X$, the *alpha-limit set* of z , denoted by $\alpha(z)$, consists of those points $y \in X$ such that $y = \lim_{j \rightarrow \infty} f^{n_j}(z)$ for some sequence of integers $n_j \rightarrow \infty$. The *omega-limit set* of z , denoted by $\omega(z)$, is similarly defined for sequences $n_j \rightarrow -\infty$. If, for some point z , $\alpha(z)$ and $\omega(z)$ each consist of a single point, we say z has *converging semi-orbits under f* .

As is well-known if f is an expansive homeomorphism of a compact metric space, then the set of points having converging semi-orbits under f is a countable set (c.f. Theorem 1 in [9] or Theorem 2.2.22 in [1]). The following represents a pairwise sensitive version of this result.

Theorem 2.7 *If f is a pairwise sensitive homeomorphism with respect to μ , then the set of points with converging semi-orbits under f has μ -measure 0.*

A referee suggested a proof of this theorem as in Theorem 2.2.22 of the Aoki-Hiraide's book [1]. Nevertheless, it seems that this does not apply in the present case since the proof in [1] is based on the finiteness of the fixed point set. Instead, we present the following alternative proof.

PROOF. Denote by $\text{Fix}(f)$ and $A(f)$ the set of fixed points and points with converging semi-orbits under f respectively. Given $x, y \in X$, $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$ we define $A(x, y, n, m)$ as the set of points $z \in X$ satisfying

$$\max \left\{ d(f^i(z), x), d(f^j(z), y) \right\} \leq \frac{1}{n}, \quad \forall i \leq -m \leq m \leq j.$$

This set is clearly compact.

Let us show that there is a sequence of fixed points x_k satisfying

$$A(f) \subset \bigcap_{n \in \mathbb{N}^+} \bigcup_{k, k', m \in \mathbb{N}^+} A(x_k, x_{k'}, n). \quad (5)$$

Indeed, $\text{Fix}(f)$ is compact since f is continuous so there is a sequence of fixed points x_k which is dense in $\text{Fix}(f)$ (we shall prove that this sequence works). Take $z \in A(f)$. Then, there are fixed points x, y such that $\alpha(z) = x$ and $\omega(z) = y$. Fix $n \in \mathbb{N}^+$. As $\alpha(z) = x$ and $\omega(z) = y$ there is $m \in \mathbb{N}^+$ such that

$$\max \{ d(f^i(z), x), d(f^j(z), y) \} \leq \frac{1}{2n}$$

whenever $i \leq -m \leq m \leq j$. But $x, y \in \text{Fix}(f)$ so there are $k, k' \in \mathbb{N}^+$ such that

$$\max \{ d(x, x_k), d(y, x_{k'}) \} \leq \frac{1}{2n}.$$

Combining these inequalities with the triangle inequality we obtain

$$\max \{ d(f^i(z), x_k), d(f^j(z), x_{k'}) \} \leq \frac{1}{n}, \quad \forall i \leq -m \leq m \leq j$$

proving $z \in A(x_k, x_{k'}, n, m)$ and so (5) holds.

Now, assume by contradiction that $\mu(A(f)) > 0$. Then, (5) implies

$$\mu \left(\bigcup_{k, k', m \in \mathbb{N}^+} A(x_k, x_{k'}, n, m) \right) > 0, \quad \forall n \in \mathbb{N}^+. \quad (6)$$

Since f is pairwise sensitive we can fix a μ -expansivity constant e (c.f. Definition 2.4). Fix also $n \in \mathbb{N}^+$ such that $\frac{1}{n} \leq \frac{e}{2}$. Applying (6) to this n we can

arrange $k, k', m \in \mathbb{N}^+$ such that

$$\mu(A(x, y, n, m)) > 0,$$

where $x = x_k$ and $y = x_{k'}$. Then, since $A(x, y, n, m)$ is compact, there are $z \in A(x, y, n, m)$ and $\delta_0 > 0$ such that

$$\mu(A(x, y, n, m) \cap B[z, \delta]) > 0, \quad \forall 0 < \delta < \delta_0,$$

where $B[\cdot, \delta]$ indicates the closed δ -ball operation. As f is continuous and m fixed we can also fix $0 < \delta < \delta_0$ such that

$$d(f^i(z), f^i(w)) \leq \frac{e}{2} \text{ whenever } -m \leq i \leq m \text{ and } d(z, w) < \delta.$$

We claim that

$$A(x, y, n, m) \cap B[z, \delta] \subset \Gamma_e(z).$$

Indeed, take $w \in A(x, y, n, m) \cap B[z, \delta]$.

Since $w \in B[z, \delta]$ one has $d(z, w) < \delta$ so

$$d(f^i(w), f^i(z)) \leq e, \quad \forall -m \leq i \leq m.$$

Since $z, w \in A(x, y, n, m)$ and $\frac{1}{n} \leq \frac{e}{2}$ one has

$$d(f^i(w), f^i(z)) \leq d(f^i(w), x) + d(f^i(z), x) \leq e$$

and

$$d(f^j(w), f^j(z)) \leq d(f^j(w), y) + d(f^j(z), y) \leq e$$

for all $i \leq -m \leq m \leq j$ proving $w \in \Gamma_e(z)$. This proves the claim.

The claim implies

$$0 < \mu(A(x, y, n, m) \cap B[z, \delta]) \leq \mu(\Gamma_e(z))$$

which is absurd by Lemma 2.1 since e is a μ -expansivity constant. This ends the proof. \square

A direct corollary is the following pairwise sensitive version of Theorem 3.1 in [10]. Denote by $\text{Per}(f)$ the set of periodic points of f .

Corollary 2.8 *Every pairwise sensitive homeomorphism f is aperiodic, i.e., $\mu(\text{Per}(f)) = 0$.*

PROOF. Recalling $\text{Fix}(f) = \{x \in X : f(x) = x\}$ we have $\text{Per}(f) = \bigcup_{n \in \mathbb{N}^+} \text{Fix}(f^n)$. Now, f^n is pairwise sensitive by Proposition 2.6, and every element of $\text{Fix}(f^n)$ is a point with converging semi-orbits of f^n , thus $\mu(\text{Fix}(f^n)) = 0$ for all n by Theorem 2.7. Therefore, $\mu(\text{Per}(f)) \leq \sum_{n \in \mathbb{N}^+} \mu(\text{Fix}(f^n)) = 0$. \square

Now, we describe pairwise sensitive homeomorphisms in dimension one. To start with we prove that there are no such homeomorphisms of compact intervals.

Theorem 2.9 *There are no pairwise sensitive homeomorphisms of a compact interval.*

PROOF. Suppose by contradiction that there is a pairwise sensitive homeomorphism of a compact interval I , with respect to some Borel probability μ of I . By Proposition 2.6 we can assume that f is orientation-preserving. Since f is also continuous we have that $\text{Fix}(f) \neq \emptyset$. But such a set is also closed since f is continuous so its complement $I \setminus \text{Fix}(f)$ consists of countably many open intervals J . Since f is orientation-preserving it is clear that every point in J is a point with converging semi-orbits therefore $\mu(I \setminus \text{Fix}(f)) = 0$ by Theorem 2.7. But $\mu(\text{Fix}(f)) = 0$ by Corollary 2.8 so $\mu(I) = \mu(\text{Fix}(f)) + \mu(I \setminus \text{Fix}(f)) = 0$ contradiction. This ends the proof. \square

Let S^1 denote the unit circle. Recall that an orientation-preserving homeomorphism of the circle S^1 is *Denjoy* if it is not topologically conjugated to a rotation [4].

Theorem 2.10 *A homeomorphism of S^1 is pairwise sensitive (with respect to some Borel probability measure μ) if and only if it is Denjoy.*

PROOF. Let f be a Denjoy homeomorphism of S^1 . As is well known f has no periodic points and exhibits a unique minimal set Δ which is a Cantor set [4]. In particular, Δ is compact without isolated points thus it exhibits nonatomic Borel probability measures μ (c.f. Corollary 6.1 in [6]). On the other hand, one sees that $f|_{\Delta}$ is an expansive homeomorphism, and so, it is pairwise sensitive with respect to μ by Corollary 2.5.

Conversely, suppose that f is a pairwise sensitive homeomorphism of S^1 with respect to some Borel probability measure μ . Suppose by contradiction that it is not Denjoy. Then, either f has periodic points or is conjugated to a rotation (c.f. [4]).

In the first case we can assume by Proposition 2.6 that f has a fixed point. Then, cutting open S^1 along the fixed point we obtain a pairwise sensitive homeomorphism of a compact interval I with respect to some Borel probability measure ν . But this contradicts Theorem 2.9.

In the second case it is well-known that there are a homeomorphism ϕ and a rotation R of S^1 such that $\phi \circ f = R \circ \phi$ (c.f. [4]). Let $\phi_*(\mu)$ be the pullback measure of S^1 defined by $\phi_*(\mu)(A) = \mu(\phi^{-1}(A))$ for all borelian A . Since f is pairwise sensitive with respect to μ we have that R is pairwise sensitive with respect to $\phi_*(\mu)$. However, isometries cannot be pairwise sensitive with respect to any Borel probability measure (c.f. the remark after Theorem 2.2 in [3]). This contradiction proves the result. \square

We finish with some short applications of our results. First we observe that combining Corollary 6.1 p. 210 in [6] and Corollary 2.5 we obtain that the set of periodic points of an expansive homeomorphism of a compact metric space is countable (this was originally proved by Utz [10]). Second we obtain a probabilistic proof of a result by Jacobsen and Utz (c.f. [2], [5]).

Corollary 2.11 *There are no expansive homeomorphisms of a compact interval.*

PROOF. Suppose by contradiction that there is an expansive homeomorphism of a compact interval I . Since the Lebesgue measure Leb is nonatomic we obtain that f is pairwise sensitive with respect to Leb by Corollary 2.5. But such homeomorphisms do not exist by Theorem 2.9. \square

Recall that the *nonwandering set* of f is the set $\Omega(f)$ consisting of those points $x \in X$ for which $U \cap (\cup_{n \in \mathbb{N}^+} f^n(U)) \neq \emptyset$ for all neighborhood U of x . As is well known $\Omega(f)$ contains the support $\text{supp}(\mu)$ of every invariant Borel probability measure μ of f . Now we prove the same but for pairwise sensitive homeomorphisms in S^1 .

Lemma 2.12 *If f is a homeomorphism of S^1 which is pairwise sensitive with respect to a Borel probability μ of S^1 , then $\text{supp}(\mu) \subset \Omega(f)$.*

PROOF. Suppose by contradiction that there is $x \in \text{supp}(\mu) \setminus \Omega(f)$. By Lemma 2.1 we have that f has a μ -expansivity constant δ . Since $x \notin \Omega(f)$ we can assume that the collection of open intervals $\{f^n(B(x, \delta)) : n \in \mathbb{Z}\}$ is disjoint. Therefore, there is $N \in \mathbb{N}$ such that the length of $f^n(B(x, \delta))$ is less than δ for $|n| \geq N$. From this and the continuity of f we can arrange $\epsilon > 0$

such that $B(x, \epsilon) \subset \Gamma_\delta(x)$ therefore $\mu(\Gamma_\delta(x)) \geq \mu(B(x, \epsilon)) > 0$ as $x \in \text{supp}(\mu)$. This contradicts Lemma 2.1 and the result follows. \square

We use this lemma together with Theorem 2.10 to obtain a probabilistic proof of the following result also by Jacobsen and Utz [5].

Corollary 2.13 *There are no expansive homeomorphisms of S^1 .*

PROOF. Suppose by contradiction that there is an expansive homeomorphism of S^1 . Since the Lebesgue measure Leb of S^1 is nonatomic we obtain from Corollary 2.5 that f is pairwise expansive with respect to Leb and so $\text{supp}(Leb) \subset \Omega(f)$ by Lemma 2.12. But, $\Omega(f)$ is a Cantor set by Theorem 2.10 and $\text{supp}(Leb) = S^1$ thus we obtain a contradiction. This ends the proof. \square

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